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On Different Modes of constructing Tables of the Values of Policies.

By HENRY W. MANLY, *of the London and Provincial Law Assurance Society.*

[Read before the Institute, 26th April, 1869.]

IN the construction of the Tables of Policy Values for the Messenger Prize Essay (1868), the operations were considerably facilitated by the use of different modes of working applicable to the different data at hand; an explanation of which forms the subject of the present paper.

In the formation of Tables the great objects to be attained are accuracy and facility of working. These may be acquired in a high degree if it be possible by means of a simple operation to deduce each value in succession from the one which immediately precedes it, in which case the chances of error are reduced to a minimum, and the construction facilitated by the work not requiring to be checked except at distant intervals. Where, however, the operations cannot be easily performed except by the aid of logarithms, errors may sometimes occur in finding the corresponding natural numbers, which will of course require checking. But although there is no way of shortening this portion of the work, still greater accuracy may be secured by the use of two different forms of tables, that is, by using a table of Antilogarithms first and checking the work by a table of Logarithms used inversely, or *vice versâ*.

Now there exists a relation between the values of policies which will be found very useful in constructing a table, namely: that

$$\frac{1-\mathrm{V}_{x|n+1}}{1-\mathrm{V}_{x|n}}=\frac{\mathbf{a}_{x+n+1}}{\mathbf{a}_{x+n}}=1-\mathrm{V}_{x+n|1}$$

or, using logarithms,

$$\lambda(1-\mathrm{V}_{x|n+1})-\lambda(1-\mathrm{V}_{x|n})=\Delta\lambda\mathbf{a}_{x+n};$$

and consequently

$$\lambda(1-\mathrm{V}_{x|n+1})=\lambda(1-\mathrm{V}_{x|n})+\Delta\lambda\mathbf{a}_{x+n}$$

from which it will be easily seen that a table of the values of $\lambda(1-\mathrm{V}_{x|n})$ can be formed by continuously adding the successive values of $\Delta\lambda\mathbf{a}_x$. This operation is best performed by means of a perforated card, cut so as to allow of 3 lines being left between each value, as represented in the annexed diagram; where the values are taken, as they will be throughout this paper, from Dr. Farr's English Life Tables No. 3 (Males, 3 per cent), because they contain most of the information required.

		$\Delta\lambda\mathbf{a}_x$
	15	·9958737
$\lambda(1-\mathrm{V}_{15\vert 1})$		·9958737
	16	·9958864
$\lambda(1-\mathrm{V}_{15\vert 2})$		·9917601
	17	·9959474
$\lambda(1-\mathrm{V}_{15\vert 3})$		·9877075
	18	·9960447
$\lambda(1-\mathrm{V}_{15\vert 4})$		·9837522

An easier process than the above would perhaps be to commence at the oldest age and add the successive values of $-\Delta\lambda\mathbf{a}_x$, since $\lambda(1-\mathrm{V}_{x|n})=\lambda(1-\mathrm{V}_{x|n+1})-\Delta\lambda\mathbf{a}_{x+n}$; thus,

$\lambda(1-V_{15\rvert 5})$		·9799198
	Ar. Co. $\Delta\lambda\mathbf{a}$ 19	·0038324
$\lambda(1-V_{15\rvert 4})$		·9837522
	18	·0039553
$\lambda(1-V_{15\rvert 3})$		·9877075
	17	·0040526
$\lambda(1-V_{15\rvert 2})$		·9917601
	16	·0041136
$\lambda(1-V_{15\rvert 1})$		·9958737

It will thus be seen that with a set of perforated cards containing the values of $\Delta\lambda\mathbf{a}_x$ (or $-\Delta\lambda\mathbf{a}_x$) the principal portion of the work will be easily performed.

The cards, however, take some little time to make and are rather awkward to use, so that something both in time and convenience would be gained if a method could be devised which would obviate the necessity of their use and at the same time be equally efficacious.

In the present instance this can be easily done, and, I think, with great advantage, from another relation of the Policy Values. Thus, we have

$$\lambda(1-V_{x\rvert n})=\lambda\mathbf{a}_{x+n}-\lambda\mathbf{a}_x$$

and

$$\lambda(1-V_{x+1\rvert n-1})=\lambda\mathbf{a}_{x+n}-\lambda\mathbf{a}_{x+1}$$
$$=\lambda(1-V_{x\rvert n})-\Delta\lambda\mathbf{a}_x$$

so that when the first column containing the values of $\lambda(1-V_{x\rvert n})$ is formed for all values of *n*, the next column, which will consist of the values of $\lambda(1-V_{x+1\rvert n})$, will be formed by the simple addition of the arithmetical complement of $\Delta\lambda\mathbf{a}_x$ to all the values in the first column, the third column would in the same manner be formed by the constant addition of ar. co. $\Delta\lambda\mathbf{a}_{x+1}$ to each value in the second, and so on throughout the Table. By this method the work will check itself in horizontal lines, the top value in each column becoming zero in the next, if all the values in that line are correct, since

$$\lambda(1-\mathrm{V}_{x|1})-\Delta\lambda\mathbf{a}_x=\lambda(1-\mathrm{V}_{x+1|0})=0.$$

The whole operation will however be best explained by an example. A preliminary Table will have to be formed of the values of $\lambda\mathbf{a}_x$ and their differences. Let us take the following from the English Life Table No. 3 (Males, 3 per cent).

Age (x).	log $\mathbf{a}_x$	Δ log $\mathbf{a}_x$	Ar. Co. Δ log $\mathbf{a}_x$
15	1·3637102	·9958737	·0041263
16	1·3595839	·9958864	·0041136
17	1·3554703	·9959474	·0040526
18	1·3514177	·9960447	·0039553
19	1·3474624	·9961676	·0038324
20	1·3436300		

The first column containing the values of $\lambda(1-\mathrm{V}_{15|n})$ will be found by placing the successive values of $\Delta\lambda\mathbf{a}_x$ on the paper, leaving 3 lines between each, and continuously adding as explained by the card method; but it will be quite sufficient to place the value of $\lambda\mathbf{a}_{15}$ at the top of a card, and, sliding it down, subtract that value from the successive values of $\lambda\mathbf{a}_x$, placing the results down in a column on ordinary cross-ruled paper, with 3 lines interval between each. It would perhaps be better to check these values at first, but it is not necessary, since, if there be any mistake made, it is sure to appear afterwards, as will be directly seen from the example, in which I have purposely made an error. We next place the value ar. co. $\Delta\lambda\mathbf{a}_{15}$ upon the top of a card, and add it to each value in the first column, placing the results immediately opposite in the second column. The first result being zero shows that the top value in the first column is correct. The third column is formed in the same manner by adding ar. co. $\Delta\lambda\mathbf{a}_{16}$ to each value in the second column; and so on throughout the Table. I may here mention what perhaps has occurred to many before me that when we have a series of values to add in this manner, the best way is to place them upon a slip of cross-ruled paper, on every other line, so that as soon as you have finished with one value you can double it under, and the next will be immediately ready for use. The annexed diagram will, I think, fully explain my meaning:—in which the lines represent where the paper is doubled under.

x	ar. co. $\Delta\lambda\mathbf{a}_x$
15	0041263
16	0041136
17	0040526
18	0039553
19	0038324

It will be observed that this method is not attended with the difficulties of adding two quantities placed horizontally to each other, for here the values to be added are placed one under the other, and their sum only withdrawn horizontally.

We will now suppose that the work has proceeded very steadily until column ($x=18$) is finished, when we find on proceeding to the formation of the next that the addition of ar. co. $\Delta\lambda\mathbf{a}_{18}$ to the first value (in the preceding column) does not make 0,—that there is in fact a difference of unity in the sixth place. We know that the error must be in the same horizontal line, because each value is formed from the one adjacent in the preceding column; and a little reflection will show that the values in that line are the differences of $\log\mathbf{a}_{15}$, $\log\mathbf{a}_{16}$, &c. subtracted from $\log\mathbf{a}_{19}$; so in order to find the error, we place $\lambda\mathbf{a}_{19}$ upon the bottom of a card and proceeding backwards subtract the values of $\lambda\mathbf{a}_{18}$, $\lambda\mathbf{a}_{17}$, &c. from it until the error is found, which in this case is in the first column.

When this part of the work is finished, all that is required is to find the natural numbers corresponding to the logarithms and subtract them from unity.

n	$x=15$	n	$x=16$	n	$x=17$	n	$x=18$	n	$x=19$
1	9958737 990544 009456								
2	9917601 981206 018794	1	9958864 990573 009427						
3	9877075 972092 027908	2	9918338 981372 018628	1	9959474 990712 009288				
4	$98375\overset{2}{\cancel{3}}2$ 963279 036721	3	$98787\overset{8}{\cancel{9}}5$ 972475 027525	2	$99199\overset{2}{\cancel{3}}1$ 981730 018270	1	$99604\overset{4}{\cancel{5}}7$ 990934 009066		
5	9799198 954816 045184	4	9840461 963931 036069	3	9881597 973105 026895	2	9922123 982228 017772	1	9961676 991214 008786

Whenever a Table of Policy Values is required the annuity values will generally be calculated, and the above method will, I think, be found the best. But sometimes it may happen that the premiums only are given, as for example, when we wish to form a table from the Office premiums. In such a case the process will be very similar, since

$$\lambda(1-V_{x|n})=\lambda(P_x+d)-\lambda(P_{x+n}+d)$$

and

$$\begin{aligned}\lambda(1-V_{x+1|n-1})&=\lambda(P_{x+1}+d)-\lambda(P_{x+n}+d)\\&=\lambda(1-V_{x|n})+\Delta\lambda(P_x+d)\end{aligned}$$

The preliminary Table to be constructed would be as follows:

Age (x)	P_x	P_x+d	$\lambda(P_x+d)$	$\Delta\lambda(P_x+d)$
15	·01415407	·04328028	$\bar{2}$·6362901	·0041256
16	·01456717	·04369338	·6404157	·0041141
17	·01498305	·04410926	·6445298	·0040531
18	·01539663	·04452284	·6485829	·0039553
19	·01580397	·04493018	·6525382	·0038322
20	·01620219	·04532840	·6563704	

The first column containing the values of $\lambda(1-V_{15|n})$ would be formed by placing $\lambda(P_{15}+d)$ at the bottom of a card and, sliding it down, subtracting the successive values of $\lambda(P_x+d)$ from it. The second column would be formed by the addition of $\Delta\lambda(P_{15}+d)$ to the values in the first column, and so on throughout the table as explained with the annuities-due. The check upon the work will be the same, and the mode of finding any error very similar; that is to say, if it occurred in the same place as above, we should place the value $\lambda(P_{19}-d)$ on the top of a card and proceeding backwards deduct it from the values $\lambda(P_{18}-d)$, $\lambda(P_{17}-d)$, &c. until the mistake was found. The results if worked out from the above values would be very similar to the above table formed from the annuities.

It will no doubt be noticed that I have here taken the values of P_x to a great many decimal places, but that was simply in order to show how closely the values by the two different methods might be made to approach. In fact the results, theoretically speaking, should be exactly the same, but, if only five or even six places had been taken instead of eight, there would have been a discrepancy between the final results by the two methods, varying to the extent sometimes of two in the fifth place.

The third and last method is one which will perhaps be the least needed of all, and has reference more especially to the case where the Office in valuing the policies credits itself with the present value of the loaded premiums, or with premiums other than the pure premiums of the Table of Mortality used in the valuation.

Here we are reduced to the formation of our Table from the familiar formula

$$V_{x|n}=A_{x+n}-P_x\mathbf{a}_{x+n};$$

but even here we shall be able to facilitate the operations very considerably, since

$$\lambda(A_{x+n}-V_{x|n})=\lambda P_x+\lambda\mathbf{a}_{x+n}$$

and

$$\begin{aligned}\lambda(A_{x+n}-V_{x+1|n-1})&=\lambda P_{x+1}+\lambda\mathbf{a}_{x+n}\\&=\lambda(A_{x+n}-V_{x|n})+\Delta\lambda P_x\end{aligned}$$

so that the successive columns of primary values will be formed by the simple addition of $\Delta\lambda P_x$.

The process to be pursued under this method will be best explained by expressing the Table of values symbolically, thus,

n	$x=15$	n	$x=16$	n	$x=17$
1	$A_{16}-P_{15}\mathbf{a}_{16}$				
2	$A_{17}-P_{15}\mathbf{a}_{17}$	1	$A_{17}-P_{16}\mathbf{a}_{17}$		
3	$A_{18}-P_{15}\mathbf{a}_{18}$	2	$A_{18}-P_{16}\mathbf{a}_{18}$	1	$A_{18}-P_{17}\mathbf{a}_{18}$

where it will be seen that the premiums are the same throughout each column, and that the assurance and annuity-due values are the same throughout each horizontal line.

The preliminary table required by this method to form the same values as before, will be

Age (x)	A_x	$\lambda\mathbf{a}_x$	λP_x	$\Delta\lambda P_x$
15	·327033	1·3637102	2·1508814	·0124938
16	·333395	·3595839	·1633752	·0122250
17	·339680	·3554703	·1756002	·0118253
18	·345815	·3514177	·1874255	·0113408
19	·351746	·3474624	·1987663	·0108077
20	·357441	·3436300	·2095740	

The first column containing the values of $\lambda P_{15}\mathbf{a}_x$ is formed by placing λP_{15} upon the top of a card, and, commencing with $\lambda\mathbf{a}_{16}$, adding it to the successive values of $\lambda\mathbf{a}_x$, placing the results down with an interval of three lines space between each. The second column is formed by adding $\Delta\lambda P_{15}$ to the values in the first, and the third by adding $\Delta\lambda P_{16}$ to those in the second, and so on throughout the Table. The check upon the work by this method is not quite so simple as by the other methods, since the first values in each column must be checked by the actual addition of λP_x to λa_{x+1}; but the error, when there is one, will be found as before in the horizontal line.

The next process is to find the corresponding natural numbers; which, having been checked, have to be subtracted from the respective values of A_x. As each line treats with only one value of A_x at a time, we require the means of moving each value across the paper in such a manner that it will serve the purpose of a minuend to all the quantities to be subtracted from it. This is most conveniently accomplished by placing upon a slip of cross-ruled paper, on every alternate line, the values of A_x, commencing at the oldest age, so that as each value is finished with, it may be doubled under, and the next be immediately ready for use. Each of the results will of course require checking, but I need hardly say that this will be accomplished by adding the last value found to the previous subtrahend, which should agree with the respective values of A_x.

n	$x=15$	n	$x=16$	n	$x=17$	n	$x=18$	n	$x=19$
1	5104653 323941 009454								
2	5063517 320887 018793	1	5188455 330252 009428						
3	5022991 317906 027909	2	5147929 327185 018630	1	5270179 336525 009290				
4	4983438 315024 036722	3	5108376 324218 027528	2	5230626 333474 018272	1	5348879 342679 009067		
5	4945114 312256 045185	4	5070052 321370 036071	3	5192302 330545 026896	2	5310555 339669 017772	1	5423963 348655 008786

I trust I have been able to make my explanations intelligible.

A Table of Policy Values may sometimes be of great utility, such as when it is desired in the investigations to value each policy separately, or when the profits have to be distributed according to the values of the policies, or even for the purpose of calculating the surrender values; and should such a table ever be required, I think I have shown that, far from the computations being difficult and laborious, they can be constructed with the greatest ease and accuracy.

The above paper was written before I had seen Dr. Zillmer's ingenious application of M. Thomas' Arithmometer to the construction of Tables of Policy Values. By means of that instrument such tables are more easily constructed and are less liable to error; but without it, his formula must be reduced to that given above before it is applicable for ordinary working.

Notes on Newton's Formulæ for Interpolation. By PROFESSOR LUDVIG OPPERMANN, *of Copenhagen.*

II.

IN the first note I gave the demonstration of Newton's formulæ of Interpolation.

Now the question arises, "How has Newton found this general solution of the problem," or rather, "how has this general form of the problem and the consequent general solution been suggested to Newton?"

The answer is given in the *Methodus Differentialis.*

There Newton proceeds as follows (in the notation I have taken the liberty of making some slight alterations):

Given, the $(n+1)$ arguments $a, b, c, d, e, f, g \ldots .$; and the corresponding $(n+1)$ values A, B, C, D, E, F, G . . . .

Proposed, to determine the algebraic integer and rational function of the nth degree, which assigns the given values to the given arguments.

Solution. Let the function in question be denoted by

$$X = \kappa_0 + \kappa_1 x + \kappa_2 x^2 + \kappa_3 x^3 + \ldots\ldots + \kappa_n x^n$$

then the $(n+1)$ quantities $\kappa_0, \kappa_1, \kappa_2, \kappa_3 \ldots . \kappa_n$ are to be determined by $(n+1)$ linear equations of the form

$$\begin{aligned} A &= \kappa_0 + \kappa_1 a + \kappa_2 a^2 + \kappa_3 a^3 + \kappa_4 a^4 + \ldots . \\ B &= \kappa_0 + \kappa_1 b + \kappa_2 b^2 + \kappa_3 b^3 + \kappa_4 b^4 + \ldots . \\ C &= \kappa_0 + \kappa_1 c + \kappa_2 c^2 + \kappa_3 c^3 + \kappa_4 c^4 + \ldots . \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$